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Spinor representation of an electromagnetic plane wave

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Abstract
The amplitude, phase and state of polarization of an electromagnetic monochromatic plane wave is expressed in terms of a two-component ($SU(2)$) spinor, which can be represented by a tangent vector to the Poincaré sphere. It is shown that the Hermitian interior product between spinors involves the parallel transport of tangent vectors along the geodesics of the sphere and that two waves are in phase, according to Pancharatnam's definition, when the tangent vectors to the sphere representing the two waves are parallel to each other along the great circle arc joining the corresponding points of the sphere.

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1. Introduction

Many years ago, Pancharatnam [1] studied the superposition of light beams, making use of the fact that the state of polarization of a light beam can be represented by a point of the Poincaré sphere. According to Pancharatnam's definition, two elliptically polarized beams are in phase when the intensity of their superposition has its maximum possible value. Then, when a given beam is decomposed as the superposition of two elliptically polarized beams, the relative phase of these two beams is related to the area of the spherical triangle on the Poincaré sphere whose vertices are the points corresponding to the two beams and the point diametrically opposite to the point corresponding to the original beam.

As pointed out in [2], the results of Pancharatnam [1] can be related to the notion of parallel transport and Pancharatnam's phase is somewhat similar to the adiabatic phase of quantum mechanics. However, there is a fundamental difference between the phases appearing in these two contexts. The adiabatic phase in quantum mechanics arises when one considers a slow change in the Hamiltonian, and this change can be represented by a curve in the parameter space of the Hamiltonian; on the other hand, when two polarized beams are superposed, the intensity of the resulting beam can be related to the geodesic of the Poincaré sphere joining the points corresponding to the polarization states of the two beams being superposed, but this geodesic is just an auxiliary object, not the curve corresponding to an actual process leading from one polarization state to the other.

The aim of this paper is to show that the results of [1] can be readily obtained by representing a polarized beam by a two-component spinor. The possibility of associating a two-component spinor with a polarized beam is present in [2]; however, in [2] no meaning is assigned to the overall complex factor of the spinor (in other words, only the ratio of the components of the spinor, $e^{-i\phi}\cot\frac{1}{2}\theta$, is associated with a point of Poincaré's sphere). As we show below, the phase of this overall factor defines the direction of a tangent vector to the unit sphere and these tangent vectors allow us to speak naturally about the concept of parallel transport.

In section 2 we present the required formalism, starting from the basic notions about two-component spinors found in elementary quantum mechanics textbooks (for a more detailed treatment, see, e.g., [3]). As we shall show, the parallel transport of vectors tangent to a sphere is always present in the interior product of two-component $SU(2)$ spinors. In section 3, we show that the components of a monochromatic plane wave can be explicitly written in a convenient way in terms of a two-component spinor that fully determines the amplitude, polarization and phase of the wave, in such a way that the intensity of the wave is proportional to the interior product of the spinor with itself. We also show that if two waves are in phase, according to Pancharatnam's definition, the angles made by the tangent vectors to the Poincaré sphere representing the waves with the great circle arc joining the corresponding points of the sphere, do coincide.

2. Two-component spinors

A two-component spinor

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \tag{1}$$

(with $\psi^1, \psi^2 \in \mathbb{C}$) defines a real vector $\mathbf{R}_\psi = (R_1, R_2, R_3)$, with

$$R_i = \psi^\dagger \sigma_i \psi \tag{2}$$

($i = 1, 2, 3$), where σ_i are the Pauli matrices, and a complex vector $\mathbf{M}_\psi = (M_1, M_2, M_3)$, with

$$M_i = \psi^{\mathrm{t}} \varepsilon \sigma_i \psi, \tag{3}$$

where

$$\varepsilon \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

and ψ^{t} is the transpose of ψ. An explicit computation shows that $\{\mathrm{Re}\,\mathbf{M}_\psi, \mathrm{Im}\,\mathbf{M}_\psi, \mathbf{R}_\psi\}$ is a right-handed orthogonal basis with $|\mathrm{Re}\,\mathbf{M}_\psi|^2 = |\mathrm{Im}\,\mathbf{M}_\psi|^2 = |\mathbf{R}_\psi|^2 = \psi^\dagger\psi$ (see equations (6) and (7)). Note that the spinors ψ and $-\psi$ define the same vectors: $\mathbf{R}_{-\psi} = \mathbf{R}_\psi$ and $\mathbf{M}_{-\psi} = \mathbf{M}_\psi$.

For any given rotation in three dimensions, there exists a matrix $U \in SU(2)$, defined up to sign, such that the ordered set $\{\mathrm{Re}\,\mathbf{M}_{U\psi}, \mathrm{Im}\,\mathbf{M}_{U\psi}, \mathbf{R}_{U\psi}\}$ coincides with the ordered set formed by the images under the rotation of $\{\mathrm{Re}\,\mathbf{M}_\psi, \mathrm{Im}\,\mathbf{M}_\psi, \mathbf{R}_\psi\}$. (This corresponds to the well-known homomorphism between $SO(3)$ and $SU(2)$.)

Parameterizing the components of ψ in the form

$$\psi = \sqrt{r}\mathrm{e}^{-\mathrm{i}\chi/2}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\phi/2}\cos\frac{1}{2}\theta\\ \mathrm{e}^{\mathrm{i}\phi/2}\sin\frac{1}{2}\theta\end{pmatrix}, \tag{5}$$

one readily finds that

$$\mathbf{R}_\psi = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = r\mathbf{e}_r \tag{6}$$

and

$$\begin{aligned}\mathbf{M}_\psi &= r\mathrm{e}^{-\mathrm{i}\chi}[(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) + \mathrm{i}(-\sin\phi, \cos\phi, 0)]\\ &= r\mathrm{e}^{-\mathrm{i}\chi}(\mathbf{e}_\theta + \mathrm{i}\mathbf{e}_\phi)\\ &= r[(\cos\chi\,\mathbf{e}_\theta + \sin\chi\,\mathbf{e}_\phi) + \mathrm{i}(-\sin\chi\,\mathbf{e}_\theta + \cos\chi\,\mathbf{e}_\phi)],\end{aligned} \tag{7}$$

where $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ is the orthonormal basis induced by the spherical coordinates.

Since $\mathrm{Re}\,\mathbf{M}_\psi$ and $\mathrm{Im}\,\mathbf{M}_\psi$ are orthogonal to $\mathbf{R}_\psi$, the vectors $\mathrm{Re}\,\mathbf{M}_\psi$ and $\mathrm{Im}\,\mathbf{M}_\psi$ can be regarded as tangent vectors at the point $\mathbf{R}_\psi$ to the sphere of radius r centered at the origin. Equations (7) show that the vectors $\mathrm{Re}\,\mathbf{M}_\psi$ and $\mathrm{Im}\,\mathbf{M}_\psi$ are obtained by rotating the orthogonal vectors $r\mathbf{e}_\theta$ and $r\mathbf{e}_\phi$ through an angle χ.

Owing to the fact that $\mathrm{Im}\,\mathbf{M}_\psi$ is determined by $\mathbf{R}_\psi$ and $\mathrm{Re}\,\mathbf{M}_\psi$ (namely, $\mathrm{Im}\,\mathbf{M}_\psi = \mathbf{R}_\psi \times \mathrm{Re}\,\mathbf{M}_\psi/|\mathbf{R}_\psi|$), a nonzero spinor ψ can be represented geometrically by a tangent vector ($\mathrm{Re}\,\mathbf{M}_\psi$) at a point of a sphere (given by $\mathbf{R}_\psi$). (Equivalently, ψ is represented by a flag [4]; the flagpole is $\mathbf{R}_\psi$ and the flag lies on the plane spanned by $\mathbf{R}_\psi$ and $\mathrm{Re}\,\mathbf{M}_\psi$.)

Apart from the (usual) Hermitian interior product between spinors, $\alpha^\dagger\beta$, which is invariant under $SU(2)$ transformations, there is an antisymmetric bilinear interior product given by $\alpha^\mathrm{t}\varepsilon\beta$, which is also invariant under $SU(2)$ transformations. These two interior products are related in the following manner. Since $\varepsilon^\mathrm{t} = -\varepsilon$ and $\varepsilon^2 = -1$, we have

$$\alpha^\dagger\beta = \overline{\alpha}^\mathrm{t}\beta = \overline{\alpha}^\mathrm{t}\varepsilon^\mathrm{t}\varepsilon\beta = (\varepsilon\overline{\alpha})^\mathrm{t}\varepsilon\beta,$$

hence, by defining the mate (or conjugate), $\hat{\alpha}$, of the two-component spinor α by

$$\hat{\alpha} \equiv -\varepsilon\overline{\alpha}, \tag{8}$$

we find that

$$\alpha^\dagger\beta = -\hat{\alpha}^\mathrm{t}\varepsilon\beta \qquad \text{and} \qquad \hat{\hat{\alpha}} = -\alpha. \tag{9}$$

Hence,

$$\hat{\alpha}^\dagger\beta = \alpha^\mathrm{t}\varepsilon\beta \tag{10}$$

and

$$\hat{\alpha}^\dagger\alpha = 0. \tag{11}$$

In this sense, $\hat{\alpha}$ is orthogonal to α. (The minus sign in definition (8) is included so that the mate of $\binom{1}{0}$ is $\binom{0}{1}$.) It may be noted that the components of the complex vector $\mathbf{M}$, defined by equation (3), can be expressed in the form $M_i = \hat{\psi}^\dagger\sigma_i\psi$ (cf equation (2)).

The mate of ψ is another spinor (which transforms in the same way as ψ under $SU(2)$), and in the case of spinor (5) one finds that

$$\hat{\psi} = \sqrt{r}\mathrm{e}^{\mathrm{i}\chi/2}\begin{pmatrix}-\mathrm{e}^{-\mathrm{i}\phi/2}\sin\frac{1}{2}\theta\\ \mathrm{e}^{\mathrm{i}\phi/2}\cos\frac{1}{2}\theta\end{pmatrix} = \sqrt{r}\mathrm{e}^{-\mathrm{i}(\pi-\chi)/2}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}(\phi+\pi)/2}\cos\frac{1}{2}(\pi-\theta)\\ \mathrm{e}^{\mathrm{i}(\phi+\pi)/2}\sin\frac{1}{2}(\pi-\theta)\end{pmatrix}, \tag{12}$$

which is of form (5) with (ϕ, θ, χ) replaced by $(\phi+\pi, \pi-\theta, \pi-\chi)$; therefore, $\mathbf{R}_{\hat{\psi}} = -\mathbf{R}_\psi$ and $\mathbf{M}_{\hat{\psi}} = -\overline{\mathbf{M}_\psi}$.

From equation (9) one finds that $\alpha^\dagger\beta = 0$ if and only if β is proportional to $\hat{\alpha}$ (or, equivalently, α is proportional to $\hat{\beta}$), which means that $\mathbf{R}_\alpha$ is antiparallel to $\mathbf{R}_\beta$. Owing to the antisymmetry of ε, the set $\{\alpha, \beta\}$ is linearly independent if and only if $\alpha^{\rm t}\varepsilon\beta \neq 0$ or, equivalently, if and only if $\hat{\alpha}^\dagger\beta \neq 0$. In particular, since for any $\alpha \neq 0, \alpha^\dagger\alpha \neq 0$, it follows that for any $\alpha \neq 0$, $\{\alpha, \hat{\alpha}\}$ is linearly independent (and, hence, an orthogonal basis for the two-component spinors).

Given a linearly independent set of spinors, $\{\alpha, \beta\}$, an arbitrary spinor, ψ, can be expressed in the form

$$\psi = c_1\alpha + c_2\beta, \tag{13}$$

for some complex scalars c_1, c_2. Owing to equation (11) we see that, for instance,

$$\hat{\alpha}^\dagger\psi = c_1\hat{\alpha}^\dagger\alpha + c_2\hat{\alpha}^\dagger\beta = c_2\hat{\alpha}^\dagger\beta,$$

hence,

$$c_2 = \frac{\hat{\alpha}^\dagger\psi}{\hat{\alpha}^\dagger\beta}. \tag{14}$$

Similarly, one obtains

$$c_1 = -\frac{\hat{\beta}^\dagger\psi}{\hat{\alpha}^\dagger\beta}, \tag{15}$$

using the fact that

$$\hat{\beta}^\dagger\alpha = \beta^{\rm t}\varepsilon\alpha = (\beta^{\rm t}\varepsilon\alpha)^{\rm t} = \alpha^{\rm t}\varepsilon^{\rm t}\beta = -\alpha^{\rm t}\varepsilon\beta = -\hat{\alpha}^\dagger\beta$$

(see equation (10)). Thus, we obtain the decomposition

$$\psi = \frac{1}{\hat{\alpha}^\dagger\beta}[-(\hat{\beta}^\dagger\psi)\alpha + (\hat{\alpha}^\dagger\psi)\beta]. \tag{16}$$

In particular, if $\beta = \hat{\alpha}$, equation (16) reduces to

$$\psi = \frac{1}{\alpha^\dagger\alpha}[(\alpha^\dagger\psi)\alpha + (\hat{\alpha}^\dagger\psi)\hat{\alpha}], \tag{17}$$

which reflects the fact that $\{\alpha, \hat{\alpha}\}$ is an orthogonal basis.

2.1. Geometrical interpretation of the interior product of two spinors

Using the fact that each spinor can be represented by means of a tangent vector to a sphere, the interior product between two unit spinors can be related to some geometric properties of the corresponding tangent vectors. Furthermore, since the interior product $\alpha^\dagger\beta$ is invariant under $SU(2)$, these geometric properties must be invariant under rotations about the center of the sphere.

We shall consider two unit spinors, α, β (that is, $\alpha^\dagger\alpha = 1 = \beta^\dagger\beta$), which means that α and β are represented by tangent vectors to the unit sphere. There exists a unique $U \in SU(2)$ that maps α into $\binom{1}{0}$. Letting $\gamma \equiv U\beta$ and expressing γ in form (5), we have

$$\alpha^\dagger\beta = (U\alpha)^\dagger U\beta = \begin{pmatrix}1\\0\end{pmatrix}^\dagger {\rm e}^{-{\rm i}X/2}\begin{pmatrix}{\rm e}^{-{\rm i}\Phi/2}\cos\frac{1}{2}\Theta\\ {\rm e}^{{\rm i}\Phi/2}\sin\frac{1}{2}\Theta\end{pmatrix} = {\rm e}^{-{\rm i}(X+\Phi)/2}\cos\tfrac{1}{2}\Theta, \tag{18}$$

where Θ, Φ are the spherical coordinates of the point of the sphere corresponding to γ (see equation (6)). Since the vector $\mathbf{R}$ corresponding to the spinor $\binom{1}{0}$ is $(0, 0, 1)$, the north pole of the sphere, the angle Θ is equal to the angle between the points of the sphere corresponding

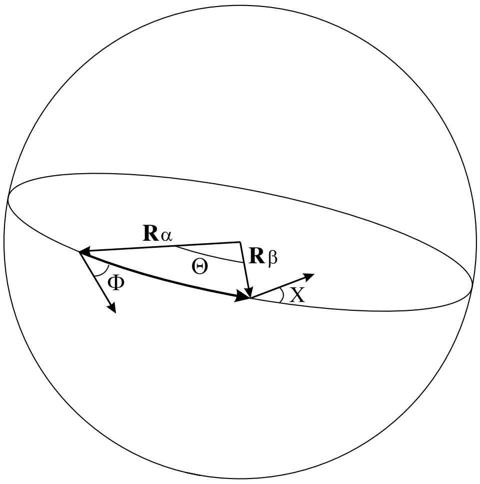

Figure 1. The unit spinors α and β are represented by tangent vectors to the unit sphere at the points $\mathbf{R}_\alpha$ and $\mathbf{R}_\beta$, respectively. The modulus of the interior product $\alpha^\dagger\beta$ depends on the angle Θ between $\mathbf{R}_\alpha$ and $\mathbf{R}_\beta$, and its phase is determined by the angles Φ and X made by the tangent vectors corresponding to α and β and the great circle arc that goes from $\mathbf{R}_\alpha$ to $\mathbf{R}_\beta$.

to α and β (and, hence, also equal to the distance between these points measured along the great circle arc passing through them, see figure 1). (That is, $\mathbf{R}_\alpha \cdot \mathbf{R}_\beta = \cos\Theta$ and, therefore, $1 + \mathbf{R}_\alpha \cdot \mathbf{R}_\beta = 2|\alpha^\dagger\beta|^2$. Making use of this last relation one can prove Wigner's theorem in the case of a two-dimensional Hilbert space [5].)

The vector $\mathrm{Re}\,\mathbf{M}$ defined by the spinor $\binom{1}{0}$ is equal to $(1, 0, 0)$ (see equations (7)); hence, this vector makes an angle $-\Phi$ with the tangent vector to the great circle arc joining the north pole with $\mathbf{R}_\gamma$. Therefore, the tangent vector $\mathrm{Re}\,\mathbf{M}_\alpha$ makes an angle $-\Phi$ with the tangent vector to the great circle arc that goes from $\mathbf{R}_\alpha$ to $\mathbf{R}_\beta$. Similarly, since the angle X appearing in equation (18) is the angle between $\mathrm{Re}\,\mathbf{M}_\gamma$ and $\mathbf{e}_\theta$, the tangent vector $\mathrm{Re}\,\mathbf{M}_\beta$ makes an angle equal to X with the tangent vector to the great circle arc joining $\mathbf{R}_\alpha$ and $\mathbf{R}_\beta$ (see figure 1). Thus, from equation (18), one concludes that the interior product $\alpha^\dagger\beta$ is a complex number whose modulus is the cosine of one-half of the angle between the points of the sphere corresponding to α and β, and its phase is one-half of the difference between the angles made by $\mathrm{Re}\,\mathbf{M}_\alpha$ and $\mathrm{Re}\,\mathbf{M}_\beta$ with respect to the geodesic of the sphere going from $\mathbf{R}_\alpha$ to $\mathbf{R}_\beta$.

In a similar manner, one finds that

$$\hat{\alpha}^\dagger\beta = (U\hat{\alpha})^\dagger U\beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\dagger \mathrm{e}^{-\mathrm{i}X/2} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\Phi/2} \cos\frac{1}{2}\Theta \\ \mathrm{e}^{\mathrm{i}\Phi/2} \sin\frac{1}{2}\Theta \end{pmatrix} = \mathrm{e}^{\mathrm{i}(\Phi-X)/2} \sin\tfrac{1}{2}\Theta, \tag{19}$$

where the angles Θ, Φ and X have the same meaning as above.

3. The electric field of a monochromatic plane wave

The (real) electric field of an elliptically polarized monochromatic plane wave propagating in the z-direction is of the form

$$\mathbf{E} = a\cos\left(\omega t - kz + \tfrac{1}{2}\chi\right)\mathbf{i} + b\sin\left(\omega t - kz + \tfrac{1}{2}\chi\right)\mathbf{j}, \tag{20}$$

where a, b are real constants, with $|a| \geqslant |b|$, ω and k are the angular frequency and wave number of the wave, respectively, provided that the axes of the ellipse are aligned with the coordinate axes. The factor 1/2 accompanying the phase χ is introduced for later convenience. Hence, in the general case, denoting by $\phi/2$ the angle made by the major axis of the ellipse with the x-axis, we have

$$\begin{aligned}\mathbf{E} = {} & \left[a\cos\tfrac{1}{2}\phi\cos\left(\omega t - kz + \tfrac{1}{2}\chi\right) - b\sin\tfrac{1}{2}\phi\sin\left(\omega t - kz + \tfrac{1}{2}\chi\right)\right]\mathbf{i} \\ & + \left[a\sin\tfrac{1}{2}\phi\cos\left(\omega t - kz + \tfrac{1}{2}\chi\right) + b\cos\tfrac{1}{2}\phi\sin\left(\omega t - kz + \tfrac{1}{2}\chi\right)\right]\mathbf{j}.\end{aligned} \tag{21}$$

Owing to the symmetry of the ellipse, it suffices to consider values of ϕ between 0 and 2π.

The ellipticity, b/a, can be expressed in the form

$$\frac{b}{a} = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right), \tag{22}$$

for some $\theta \in [0, \pi]$ (since $|b/a| \leqslant 1$). The ratio b/a is positive for $0 \leqslant \theta \leqslant \pi/2$ (in which case the wave has right-hand polarization) and b/a is negative for $\pi/2 \leqslant \theta \leqslant \pi$ (then the wave has left-hand polarization). In this way, $\theta = 0$ and $\theta = \pi$ correspond to circular polarization, while $\theta = \pi/2$ gives linear polarization. Making use of equation (22), equation (21) can be rewritten in the form

$$\begin{aligned}\mathbf{E} = A\big\{ & \left[\cos\tfrac{1}{2}\theta\cos\left(\omega t - kz + \tfrac{1}{2}\chi + \tfrac{1}{2}\phi\right) + \sin\tfrac{1}{2}\theta\cos\left(-\omega t + kz - \tfrac{1}{2}\chi + \tfrac{1}{2}\phi\right)\right]\mathbf{i} \\ & + \left[\cos\tfrac{1}{2}\theta\sin\left(\omega t - kz + \tfrac{1}{2}\chi + \tfrac{1}{2}\phi\right) + \sin\tfrac{1}{2}\theta\sin\left(-\omega t + kz - \tfrac{1}{2}\chi + \tfrac{1}{2}\phi\right)\right]\mathbf{j}\big\},\end{aligned} \tag{23}$$

where A is a real constant.

By considering the angles θ and ϕ as spherical coordinates in the usual manner (i.e., θ as the polar angle and ϕ as the azimuthal angle), each pair of values (θ, ϕ) defines a point of a sphere, which in this context is called Poincaré's sphere [6–8]. Hence, a point of Poincaré's sphere determines the state of polarization. A set of parameters more commonly employed to specify the polarization of a wave is given by the Stokes parameters, s_0, s_1, s_2, s_3, which are related to the angles θ and ϕ by means of [6, 7] (see also [1] and the references cited therein),

$$(s_1, s_2, s_3) = s_0(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{24}$$

where s_0 is the total flux density.

Thus, the nonzero components of the electric field are given by

$$\begin{aligned} E_x + \mathrm{i}E_y &= A\left[\cos\tfrac{1}{2}\theta\, \mathrm{e}^{\mathrm{i}(\omega t - kz + \chi/2 + \phi/2)} + \sin\tfrac{1}{2}\theta\, \mathrm{e}^{\mathrm{i}(-\omega t + kz - \chi/2 + \phi/2)}\right], \\ E_x - \mathrm{i}E_y &= A\left[\cos\tfrac{1}{2}\theta\, \mathrm{e}^{-\mathrm{i}(\omega t - kz + \chi/2 + \phi/2)} + \sin\tfrac{1}{2}\theta\, \mathrm{e}^{-\mathrm{i}(-\omega t + kz - \chi/2 + \phi/2)}\right],\end{aligned} \tag{25}$$

or, in terms of the unit two-component spinor

$$o = \begin{pmatrix} o^1 \\ o^2 \end{pmatrix} = \mathrm{e}^{-\mathrm{i}\chi/2}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2}\cos\frac{1}{2}\theta \\ \mathrm{e}^{\mathrm{i}\phi/2}\sin\frac{1}{2}\theta \end{pmatrix} \tag{26}$$

and its mate

$$\hat{o} = \begin{pmatrix} \hat{o}^1 \\ \hat{o}^2 \end{pmatrix} = \mathrm{e}^{\mathrm{i}\chi/2}\begin{pmatrix} -\mathrm{e}^{-\mathrm{i}\phi/2}\sin\frac{1}{2}\theta \\ \mathrm{e}^{\mathrm{i}\phi/2}\cos\frac{1}{2}\theta \end{pmatrix} \tag{27}$$

(cf equations (5) and (12)), we have

$$\begin{aligned} E_x + \mathrm{i}E_y &= A(\mathrm{e}^{-\mathrm{i}(\omega t - kz)}o^2 + \mathrm{e}^{\mathrm{i}(\omega t - kz)}\hat{o}^2), \\ E_x - \mathrm{i}E_y &= A(\mathrm{e}^{-\mathrm{i}(\omega t - kz)}o^1 - \mathrm{e}^{\mathrm{i}(\omega t - kz)}\hat{o}^1).\end{aligned} \tag{28}$$

Thus, all the information about the amplitude of the wave is given by A, and the polarization and phase are given by the two-component unit spinor o.

As pointed out in section 2, two unit spinors, o and $-o$, are represented by the same tangent vector to the unit sphere, but the electric field of a monochromatic plane wave changes sign when the spinor o does (see equation (28)). This behavior is directly related to the two-to-one relation between $SU(2)$ and $SO(3)$. In the present context, the action of the group $SU(2)$ on the two-component spinors corresponds to changes in the state of polarization of the wave, which are represented by rotations on the Poincaré sphere.

The time average of the square of the electric field (28) is given by

$$\langle \mathbf{E}^2 \rangle = A^2(\hat{o}^2 o^1 - \hat{o}^1 o^2) = A^2 o^\dagger o = A^2, \tag{29}$$

where we have made use of equation (9). By defining the two-component spinor ψ by

$$\psi \equiv Ao, \tag{30}$$

we can rewrite equation (28) as

$$\begin{aligned} E_x + \mathrm{i}E_y &= \mathrm{e}^{-\mathrm{i}(\omega t - kz)}\psi^2 + \mathrm{e}^{\mathrm{i}(\omega t - kz)}\hat{\psi}^2, \\ E_x - \mathrm{i}E_y &= \mathrm{e}^{-\mathrm{i}(\omega t - kz)}\psi^1 - \mathrm{e}^{\mathrm{i}(\omega t - kz)}\hat{\psi}^1 \end{aligned} \tag{31}$$

(cf [2], equation (2)). Then,

$$\langle \mathbf{E}^2 \rangle = \psi^\dagger \psi. \tag{32}$$

3.1. Superposition of monochromatic waves

Now, let us consider two monochromatic plane waves of the same frequency propagating in the same direction, corresponding to the spinors $A_1\alpha$ and $A_2\beta$, where A_1, A_2 are positive real numbers and α, β are unit spinors. From equations (28) we see that

$$\psi = A_1\alpha + A_2\beta$$

(cf equation (13)) is the spinor corresponding to the superposition of these waves. Writing $\psi = A\gamma$, where γ is a unit spinor and A is a positive real number, from equations (15) and (19) we see that

$$A_1^2 = \frac{|\hat{\beta}^\dagger\psi|^2}{|\hat{\alpha}^\dagger\beta|^2} = A^2\frac{|\hat{\beta}^\dagger\gamma|^2}{|\hat{\alpha}^\dagger\beta|^2} = A^2\frac{\sin^2\frac{1}{2}a}{\sin^2\frac{1}{2}c},$$

where a is the angle between the points of the unit sphere corresponding to β and γ and c is the angle between the points corresponding to α and β. According to equation (29), this means that the intensity, I, of the superposition of the two waves is related to the intensity, I_1, of the wave represented by $A_1\alpha$ by $I_1 = I\sin^2\frac{1}{2}a\big/\sin^2\frac{1}{2}c$.

Similarly, making use of equation (14), we obtain

$$A_2^2 = \frac{|\hat{\alpha}^\dagger\psi|^2}{|\hat{\alpha}^\dagger\beta|^2} = A^2\frac{|\hat{\alpha}^\dagger\gamma|^2}{|\hat{\alpha}^\dagger\beta|^2} = A^2\frac{\sin^2\frac{1}{2}b}{\sin^2\frac{1}{2}c},$$

where b is the angle between the points of the unit sphere corresponding to α and γ. Thus, $I_2 = I\sin^2\frac{1}{2}b\big/\sin^2\frac{1}{2}c$ (cf [1], equation (3)).

The amplitudes A_1, A_2 and A are also related by

$$A^2 = \psi^\dagger\psi = (A_1\alpha + A_2\beta)^\dagger(A_1\alpha + A_2\beta) = A_1^2 + A_2^2 + 2A_1A_2\mathrm{Re}(\alpha^\dagger\beta). \tag{33}$$

Thus, making use of equation (18), we have

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos\tfrac{1}{2}\Theta\cos\tfrac{1}{2}(X + \Phi), \tag{34}$$

which shows that, according to Pancharatnam's definition, the waves being superposed are in phase if and only if $X = -\Phi$. This means that the angles made by $\mathrm{Re}\,\mathbf{M}_\alpha$ and $\mathrm{Re}\,\mathbf{M}_\beta$ with the great circle arc joining the points of the unit sphere corresponding to α and β, coincide. In other words, $\mathrm{Re}\,\mathbf{M}_\beta$ is obtained by transporting $\mathrm{Re}\,\mathbf{M}_\alpha$ parallel to itself along the geodesic joining the corresponding points of the unit sphere. Then, the fact that the sphere has a nonzero constant curvature implies that if α is in phase with β, and β is in phase with ζ, then there is a phase difference between α and ζ equal to one-half of the area of the geodesic triangle with vertices at $\mathbf{R}_\alpha$, $\mathbf{R}_\beta$ and $\mathbf{R}_\zeta$ [2].

Equation (33) shows that there is no constructive or destructive interference between the waves if and only if $\alpha^\dagger\beta$ is equal to zero or is pure imaginary.

As shown in the preceding section, $\alpha^\dagger\beta$ is equal to zero if and only if β is proportional to $\hat{\alpha}$, which means that the points of the Poincaré sphere corresponding to α and β are diametrically opposite (this conclusion also follows from equation (18)); in this case, it is said that the two waves have opposite polarization [1].

4. Concluding remarks

Equations (24) show that in the case of a completely polarized wave, the Stokes parameters depend on three independent variables; in contrast, an arbitrary two-component spinor contains four independent real parameters that fully determine the components of the wave. Since the amplitude, polarization state and phase of a monochromatic plane electromagnetic wave is encoded in a two-component spinor, it would be interesting to find, for instance, the effect of the transmission of a wave through an anisotropic medium, which must be represented by some mapping of the spinor space into itself.

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